1. Что такое ЛП (LP). Интуиция. Идея симплекс метода
2. Реальные примеры задач ЛП
3. Mixed Integer Programming.

Introduction to Linear Programming

What is LP

$f(x) = Ax + b = \sum_i \text{хрень}_i \cdot x_i$, $x \in \mathbb{R}^n$

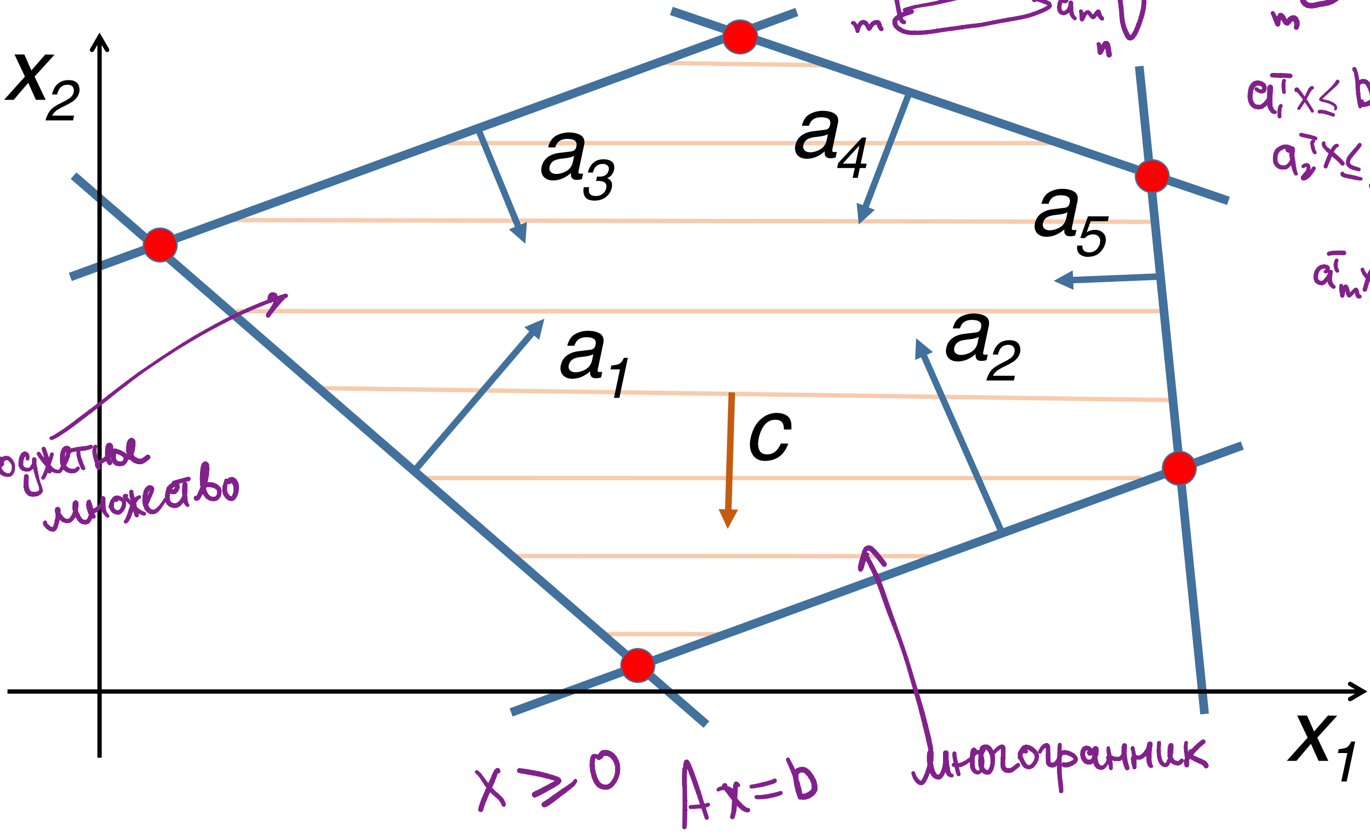
Generally speaking, all problems with linear objective and linear equalities\inequalities constraints could be considered as Linear Programming. However, there are some widely accepted formulations.

$\nabla f = c$ $\qquad$ $-\nabla f = -c$

$$
\begin{aligned}
f(x) = \min_{x \in \mathbb{R}^n} & \; c^\top x \\
\text{s.t. } & Ax \leq b
\end{aligned}
\tag{LP.Basic}
$$

$a_1, \dots, a_m$; A ($m \times n$), x (n), b (m)

$a_1^T x \leq b_1$
$a_2^T x \leq b_2$
$\vdots$
$a_m^T x \leq b_m$



for some vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$. Where the inequalities are interpreted component-wise.

нужно: $Ax = b$

есть: $\begin{cases} Ax \leq b \\ + \\ Ax \geq b \to -Ax \leq -b \end{cases}$

Standard form

This form seems to be the most intuitive and geometric in terms of visualization. Let us have vectors $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times n}$.

$$
\begin{aligned}
\min_{x \in \mathbb{R}^n} & \; c^\top x \\
\text{s.t. } & Ax = b \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{LP.Standard}
$$

Canonical form

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^n} c^\top x \\
\text{s.t. } & Ax \leq b \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
\tag{LP.Canonical}
$$

$Ax \geqslant b$

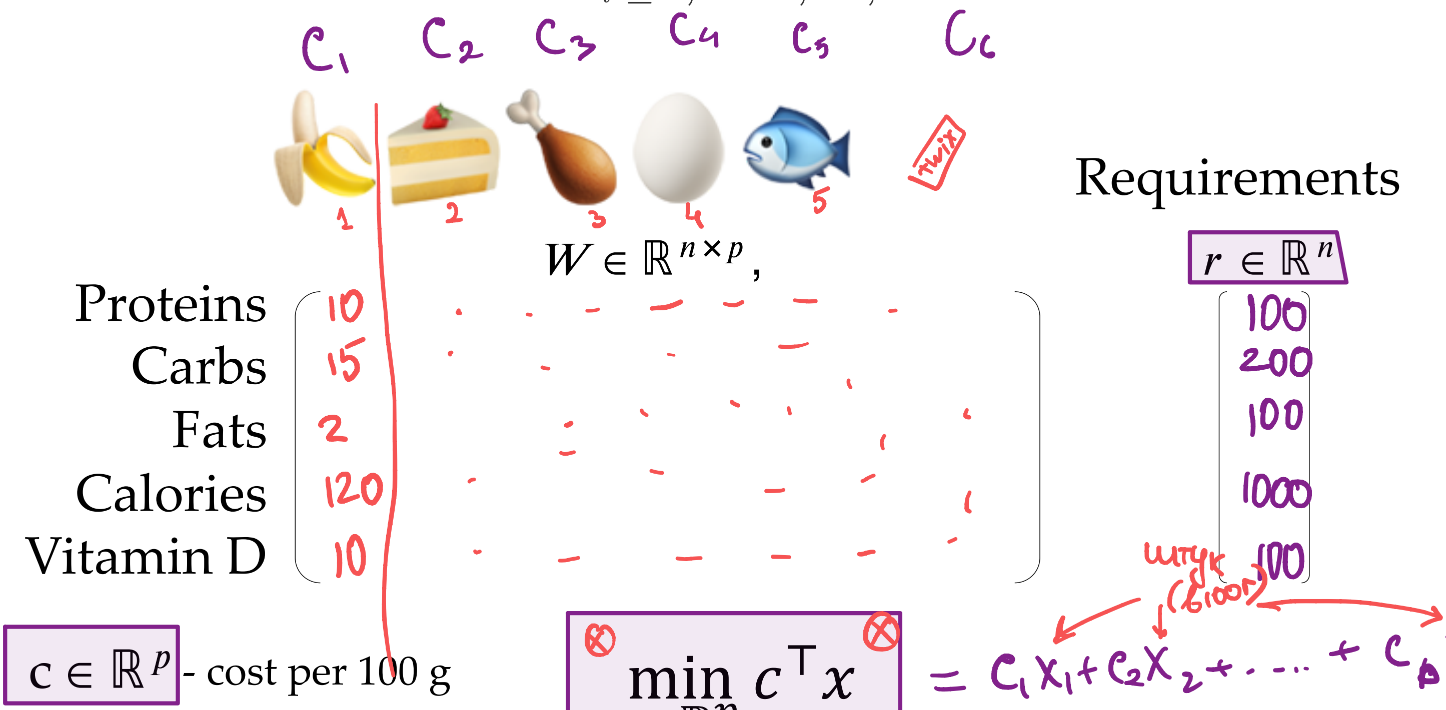
$x =$

Real world problems

Diet problem

Imagine, that you have to construct a diet plan from some set of products: 🍌🍰🍗🥚🐟. Each of the products has its own vector of nutrients. Thus, all the food information could be processed through the matrix W. Let also assume, that we have the vector of requirements for each of nutrients $r \in \mathbb{R}^n$. We need to find the cheapest configuration of the diet, which meets all the requirements:

$$
\begin{aligned}
&\min_{x \in \mathbb{R}^p} c^\top x \\
\text{s.t. } & Wx \geq r \\
& x_i \geq 0, \; i = 1, \dots, n
\end{aligned}
$$



$$\boxed{\min_{x \in \mathbb{R}^p} c^\top x \quad Wx \geq r} = c_1x_1 + c_2x_2 + \dots + c_px_p$$

штук (в 100г)

How to retrieve LP

Basic transformations

Inequality to equality by increasing the dimension of the problem by m.

$$
Ax \leq b \leftrightarrow \begin{cases} Ax + z = b \\ z \geq 0 \end{cases}
$$

unsigned variables to nonnegative variables.

$$
x \leftrightarrow \begin{cases} x = x_+ - x_- \\ x_+ \geq 0 \\ x_- \geq 0 \end{cases}
$$

Chebyshev approximation problem

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_\infty \leftrightarrow \min_{x \in \mathbb{R}^n} \max_{i} |a_i^\top x - b_i|
$$

$$
\begin{split}
&\min_{t \in \mathbb{R}, x \in \mathbb{R}^n} t \\
\text{s.t. } &a_i^\top x - b_i \leq t, \; i = 1, \dots, n \\
&-a_i^\top x + b_i \leq t, \; i = 1, \dots, n
\end{split}
$$

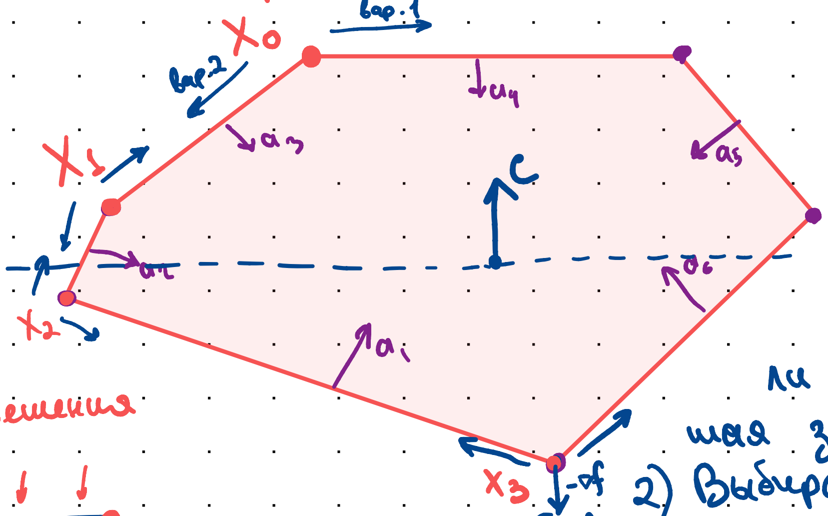
l_1 approximation problem

$$
\min_{x \in \mathbb{R}^n} \|Ax - b\|_1 \leftrightarrow \min_{x \in \mathbb{R}^n} \sum_{i=1}^n |a_i^\top x - b_i|
$$

$$
\begin{split}
&\min_{t \in \mathbb{R}^n, x \in \mathbb{R}^n} \mathbf{1}^\top t \\
\text{s.t. } &a_i^\top x - b_i \leq t_i, \; i = 1, \dots, n \\
&-a_i^\top x + b_i \leq t_i, \; i = 1, \dots, n
\end{split}
$$

Idea of simplex algorithm

Симплекс - алгоритм



min $c^T x$

$Ax \le b$

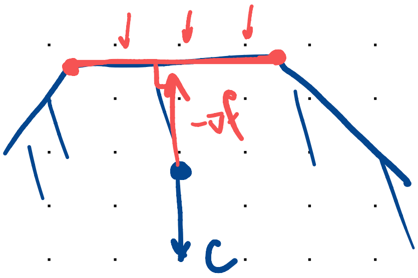
идея: решением является угловая точка

Алгоритм.

1) Находясь в угловой точке посмотреть можно ли двигаться вдоль грани, уменьшая знач. функции.

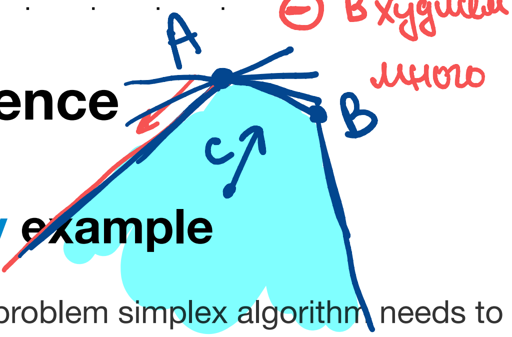
2) Выбираем направление наискор. убывания по граням.

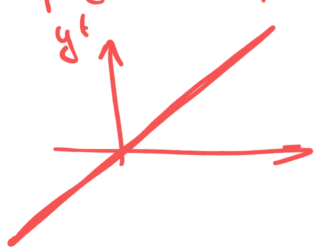
3) Если нельзя уменьшить значение $f(x)$, то FINISH.



⊖ Найти X_0 - сложно!

⊖ В худшем случае придется перебрать экспоненц. много вершин.





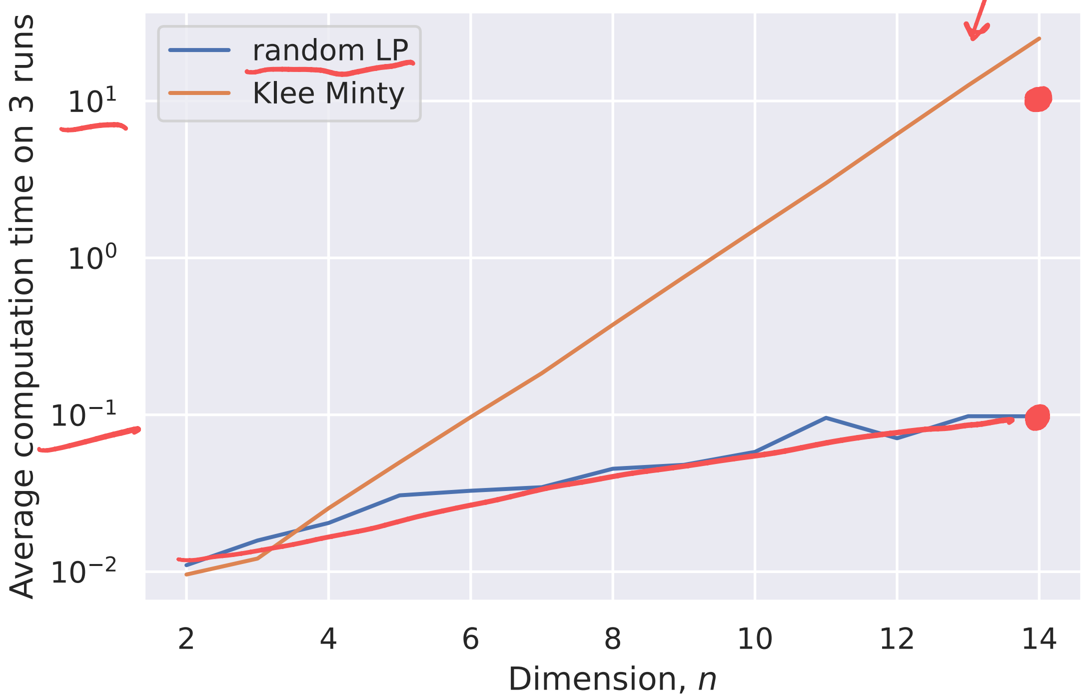
Convergence

Klee Minty example

In the following problem simplex algorithm needs to check $2^n - 1$ vertexes with $x_0 = 0$.

$$
\begin{aligned}
& \max_{x \in \mathbb{R}^n} 2^{n-1}x_1 + 2^{n-2}x_2 + \dots + 2x_{n-1} + x_n \\
\text{s.t. } & x_1 \le 5 \\
& 4x_1 + x_2 \le 25 \\
& 8x_1 + 4x_2 + x_3 \le 125 \\
& \dots \\
& 2^n x_1 + 2^{n-1}x_2 + 2^{n-2}x_3 + \ldots + x_n \le 5^n \qquad x \ge 0
\end{aligned}
$$

наихудший пример



Приложения

Summary

- A wide variety of applications could be formulated as the linear programming.
- Simplex algorithm is simple, but could work exponentially long.
- Khachiyan's ellipsoid method is the first to be proved running at polynomial complexity for LPs. However, it is usually slower than simplex in real problems.
- Interior point methods are the last word in this area. However, good implementations of simplex-based methods and interior point methods are similar for routine applications of linear programming.

10x x16

CPLEX Gurobi

1991 1998 2007 2012 t

А. Райгородский

Н. Литвак

Code

Open in Colab

Алгоритмы и software

x 29000

Hardware

x8000

x 470k

Materials

- Linear Programming. in V. Lempitsky optimization course.
- Simplex method. in V. Lempitsky optimization course.
- Overview of different LP solvers
- TED talks watching optimization
- Overview of ellipsoid method
- Comprehensive overview of linear programming
- Converting LP to a standard form

1. Решать LP современными методами на старом железе

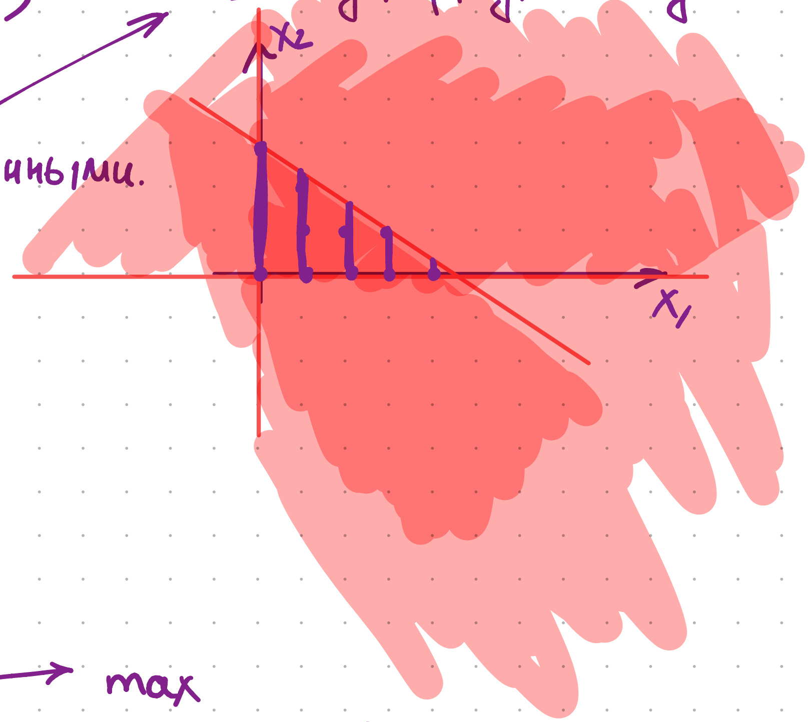
2. Решать старыми методами на новом железе.

x20

x запредельное число

Mixed Integer programming (MIP) Integer programming

все переменные должны быть целочисленными.



$x_i \in \{0,1\}$ 2^n

$n = 20-30$ уже не переберешь

!НЕ ВЫПУКЛАЯ!

Пример: $f(x) = 8x_1 + 11x_2 + 6x_3 + 4x_4 \to \max$

$5x_1 + 7x_2 + 4x_3 + 3x_4 \leq 14$

$\boxed{x_i \in \{0,1\}}$? $x_i \in [0,1]$

Оптимальное решение:

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = 0.5 \\ x_4 = 0 \end{cases} \to f = 22.$$

пусть $x_3 = 1$

$$\left.\begin{matrix} x_1 = 1 \\ x_2 = 1 \\ x_3 = 1 \\ x_4 = 0 \end{matrix}\right| \to f = 25.$$

ВНЕ БЮДЖЕТА

пусть $x_3 = 0$

$$\left.\begin{matrix} x_1 = 1 \\ x_2 = 1 \\ x_3 = 0 \\ x_4 = 0 \end{matrix}\right| \to f = 19$$

НЕ ОПТИМАЛЬНО (можно лучше)

Оптимальное решение:

$$\left.\begin{cases} x_1 = 0 \\ x_2 = 1 \\ x_3 = 1 \\ x_4 = 1 \end{cases}\right| \to f = 21$$

convex relaxation of MIP НЕ ВСЕГДА РАБОТАЕТ

КАК ~~ЖИТЬ~~ БЫТЬ ?

BRANCH AND BOUND

(метод ветвей и границ)

1) Формулируем и пытаемся решать выпуклую релаксацию задачи.

$$8x_1+11x_2+6x_3+4x_4 \to \max$$
$$5x_1+7x_2+4x_3+3x_4 \le 14$$

$$x^* = (x_1^* \dots x_n^*)$$

если все x_i^* (нужные) - целые, то

FINISH

$(1, 1, 0.5, 0)$

Как минимум, у нас есть верхняя оценка $f^* \le 22$

пробуем округлить вниз x^* и получаем нижнюю оценку $f^* \ge 19$

$19 \le f^* \le 22$

$x_3 = 0$

$$8x_1+11x_2+6x_3+4x_4 \to \max$$
$$5x_1+7x_2+4x_3+3x_4 \le 14$$
$$x_1, x_2, x_4 \in [0,1]$$
$$x_3 = 0$$

$(1, 1, 0, 0.667)$ $f = 21,65$

1 1 0 0 $f^* \ge 19$

$x_3 = 1$

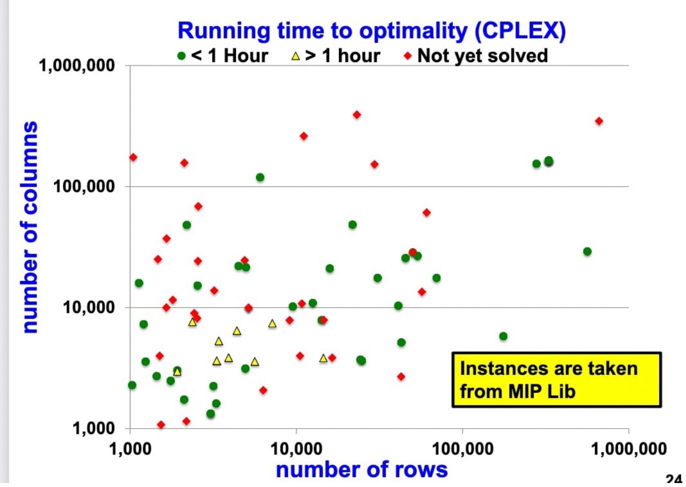
$$8x_1+11x_2+6x_3+4x_4 \to \max$$
$$5x_1+7x_2+4x_3+3x_4 \le 14$$
$$x_1, x_2, x_4 \in [0,1]$$
$$x_3 = 1$$

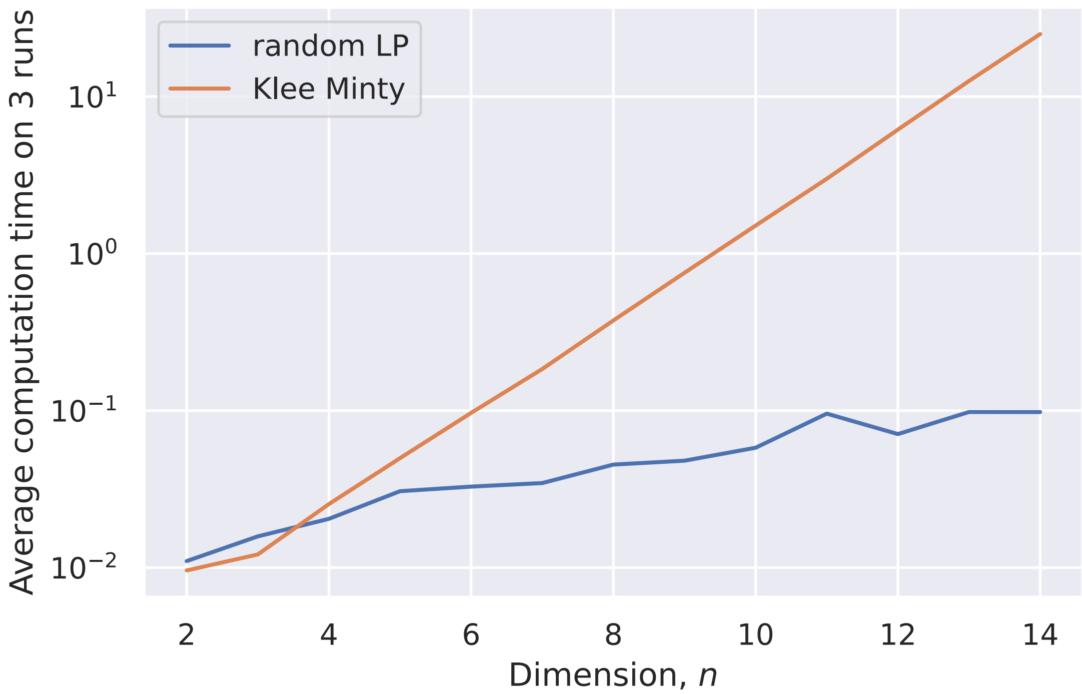
$(1, 0.714, 1, 0)$ $f = 21.85$

$f^* \ge 14$

Summary:

- заметно сложнее выпуклых задач
- выпуклая релаксация работает не всегда
- НЕ всегда понятна реальная сложность задачи ЗАРАНЕЕ





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